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Exact formulations and nearly exact numerical solutions for convection in turbulent flow between parallel plates

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Abstract

Simple but exact integral formulations are presented for the velocity and temperature distributions and for the friction factor and Nusselt number in fully developed turbulent flow and convection between parallel plates in terms of the dimensionless local turbulent shear stress and heat flux density. Essentially exact values for these quantities have been obtained by evaluating the integrals numerically for a number of special conditions for which the uncertainty of the integrands is minimal. Such values provide criteria for evaluation of experimental data and more approximate solutions, as well as a basis for the construction of generalized correlating equations.

Keywords: Turbulent flow; Turbulent convection; Fully developed convection; Numerical solutions

1. Introduction

The differential and integral formulations for forced convection in turbulent flow between parallel plates of unlimited extent are somewhat simpler than those for round tubes by virtue of the absence of curvature. Nevertheless, the behavior in this configuration has received less attention than that in round tubes and annuli because (1) it only constitutes a hypothetical limiting case for a rectangular channel of asymptotically large aspect ratio and a circular annulus with an aspect ratio approaching unity, and (2) rectangular channels and circular annuli of extreme aspect ratios have few practical applications. The principal ones involve heat (or mass) transfer from one surface to the other through a fluid stream or heat transfer from both surfaces to the fluid in compact heat exchangers. However, the latter are seldom operated in the turbulent regime owing to the large pressure drop associated with small spacings and high velocities.

Churchill and coauthors [1–7,9] have shown that fully developed flow and convection in all one-dimensional channels may be expressed in the form of simple but exact integrals of the local fractions of the shear stress and the heat flux density due to the turbulent fluctuations. Furthermore, they have derived very accurate correlating equations for the local turbulent shear stress in round tubes and parallel-plate channels. Although correlating equations of comparable accuracy and generality have not yet been achieved for the local turbulent heat flux den-

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Nomenclature

a	radius of circular tube
a^+	dimensionless radius of circular tube, $a(\tau_w\rho)^{1/2}/\mu$
b	half-spacing of parallel-plate channel
b^+	dimensionless half-spacing of parallel-plate channel, $b(\tau_w\rho)^{1/2}/\mu$
c	specific heat capacity of fluid at constant pressure
e	effective roughness of plate
f	Fanning friction factor, $2\tau_w/\rho u_m^2$
j	local heat flux density in y-direction
j_w	heat flux density from the wall at $y = 0$
k	thermal conductivity of fluid
Nu_b	Nusselt number for heat transfer between plates, $j_w b/k(T_w - T_b)$
Nu_{4b}	Nusselt number for heat transfer from both plates, $4j_w b/k(T_w - T_m)$
Pr	Prandtl number
Pr_t	turbulent Prandtl number defined by Eq. (9)
Re_b	Reynolds number based on half-spacing between plates, $bu_m\rho/\mu$
Re_{4b}	Reynolds number based on hydraulic diameter, $4bu_m\rho/\mu$
Sc	Schmidt number
T	temperature
T^+	dimensionless temperature, $k(T_w - T)(\tau_w\rho)^{1/2}/\mu j_w$
T'	fluctuation in temperature
$(\overline{T'v'})^{++}$	dimensionless heat flux density due to turbulence, $\rho c\overline{T'v'}/j$
u	velocity component in x-direction
u^+	dimensionless velocity component in x-direction, $u(\rho/\tau_w)^{1/2}$
u'	fluctuation of velocity component in x-direction
$(\overline{u'v'})^{++}$	dimensionless shear stress due to turbulence, $-\rho\overline{u'v'}/\tau$
v'	fluctuation of velocity component in y-direction
x	coordinate in direction of flow
y	distance from wall
y^+	dimensionless distance from wall, $y(\tau_w\rho)^{1/2}/\mu$
Z	fractional distance from central plane, $1-(y^+/b^+)$.

Greek symbols

γ	fractional deviation of heat flux density from linearity [see Eq. (29)]
μ	dynamic viscosity of fluid
ρ	specific density of fluid
τ	shear stress.

Subscripts

b	at central plane
m	mean value
w	at wall.

sity, the effect of such uncertainty may be avoided or minimized for some special conditions. The specific aspects of this body of prior work that are relevant to the current work may be summarized as follows. Churchill and Chan [7] first derived exact integral expressions for the turbulent flow in one-dimensional channels in terms of the ratio of the local turbulent shear stress to the local shear stress at the wall and showed that such expressions are not only simpler but also free of the intrinsic anomalies associated with the eddy-viscosity and mixing-length models. Churchill and Chan [5,6] subsequently developed a single generalized correlating equation with an asymptotic structure for the local turbulent shear stress in both round tubes and parallel-plate channels and used this expression to develop generalized and even more accurate correlating equations for the time–mean velocity distribution and the friction factor. (The non-chronological order of these references is an artifact of the process of review and publication.) Churchill [2,3] showed that the corresponding integral expressions were possible and advantageous for heat transfer and also that subtle improvements could be attained by expressing the differential and integral balances for both momentum and energy in terms of the local fraction of the transport due to turbulence rather than as a fraction of the value at the wall.

The objective of the work reported herein has been to develop improved numerical results for convection in turbulent flow corresponding to those of Heng et al. [9] for a round tube. The formulations and computed values are for fully developed convection with two thermal boundary conditions: (1) equal uniform heating of the fluid stream by both plates and (2) heat transfer through the fluid from one isothermal plate to another at a lower temperature. The latter condition is equivalent to equal uniform heating of one plate and cooling of the other.

2. Momentum transfer

The time-averaged and once-integrated differential balance for momentum in the direction away from the wall for steady fully developed isothermal flow of an incompressible Newtonian fluid between parallel plates of unlimited extent with a total spacing of $2b$ may be expressed in the following simplified form suggested by Churchill [3]:

$$\left(1-\frac{y^+}{b^+}\right)[1-(\overline{u'v'})^{++}]=\frac{du^+}{dy^+} \quad (1)$$

The superiority of Eq. (1) over prior formulations is a consequence of the use of the term $(\overline{u'v'})^{++}=-\rho\overline{u'v'}/\tau$, which has physical significance as the local fraction of the transport of momentum away from the wall due to turbulence and thereby avoids any heurism or empiricism such as that inherent in the eddy-viscosity and mixing-length models. Eq. (1) may be integrated formally to obtain the following exact expression for the time–mean velocity distribution:

$$\begin{aligned} u^+ &= \int_0^{y^+}\left(1-\frac{y^+}{b^+}\right)[1-(\overline{u'v'})^{++}]dy^+ \\ &= y^+-\frac{(y^+)^2}{2b^+}-\int_0^{y^+}\left(1-\frac{y^+}{b^+}\right)(\overline{u'v'})^{++}dy^+ \\ &= \frac{b^+}{2}\int_{Z^2}^{1}[1-(\overline{u'v'})^{++}]dZ^2 \\ &= \frac{b^+}{2}(1-Z^2)-\frac{b^+}{2}\int_{Z^2}^{1}(\overline{u'v'})^{++}\,dZ^2 \end{aligned} \quad (2)$$

Each of the four forms on the right-hand-side of Eq. (2) has its own advantages. For example, the two partially integrated forms reveal that the contribution of turbulence is simply a deduction from the well-known expressions for purely laminar flow at the same value of b^+, while the expressions in terms of $Z = 1-(y^+/b^+)$ are somewhat simpler in form than the more explicit ones in terms of y^+. It follows by means of integration by parts that:

$$\begin{aligned} \left(\frac{2}{f}\right)^{1/2} &\equiv u_m^+ \equiv \frac{1}{b^+}\int_0^{b^+}u^+dy^+ \\ &= \int_0^{b^+}\left(1-\frac{y^+}{b^+}\right)^2[1-(\overline{u'v'})^{++}]dy^+ \\ &= \frac{b^+}{3}-\int_0^{b^+}\left(1-\frac{y^+}{b^+}\right)^2(\overline{u'v'})^{++}dy^+ \end{aligned} \quad (3)$$

Churchill [4] has proposed on the basis of the recent experimental data of Zagarola [15] for the shear friction and the time–mean velocity distribution in round pipes, the following modification of the correlating equation of Churchill and Chan [6] for the turbulent shear stress in turbulent flow in a round tube:

$$(\overline{u'v'})^{++}=\left(\left[0.7\left(\frac{y^+}{10}\right)^3\right]^{-8/7}+\left|\exp\left\{\frac{-1}{0.436y^+}\right\}-\frac{1}{0.436a^+}\left(1+\frac{6.95y^+}{a^+}\right)\right|^{-8/7}\right)^{-7/8} \quad (4)$$

Eq. (4) may be inferred, on the basis of the analogy of MacLeod [12], to be valid for flow between parallel plates if b^+ is simply substituted for a^+. It is presumed on the basis of comparisons with various sets of experimental data and predicted values of u and $\overline{u'v'}$ to provide accurate predictions for $b^+ > 300$ and all y^+ [6]. The nominal limitation to $b^+ > 300$ is due to the vanishing of the "overlap" regime (the semilogarithmic regime of the velocity distribution), which appears in Eq. (4) as the exponential term.

The values of u^+ computed from Eq. (2) using $(\overline{u'v'})^{++}$ from Eq. (4) may be represented quite closely by:

$$u^+=\left\{\left[\frac{(y^+)^2}{1+y^+-\exp\left\{-\frac{7}{4}\left(\frac{y^+}{10}\right)^4\right\}}\right]^{-3}+\left[\frac{1}{0.436}\ln\{1+13.35y^+\}+6.824\left(\frac{y^+}{b^+}\right)^2-5.314\left(\frac{y^+}{b^+}\right)^3\right]^{-3}\right\}^{-1/3} \quad (5)$$

while those of u_m^+, as computed from Eq. (3) using $(\overline{u'v'})^{++}$ from Eq. (4) may be represented even more closely by:

$$\left(\frac{2}{f}\right)^{1/2}=u_m^+=4.615-\frac{155.3}{b^+}+\frac{1}{0.436}\ln\{b^+\} \quad (6)$$

The terms in $(y^+/b^+)^2$ and $(y^+/b^+)^3$ in Eq. (5) account for the contribution of the wake to the time–mean velocity distribution, while the unfamiliar, theoretically based term in $(b^+)^{-1}$ in Eq. (6) is a consequence of accounting for the decreased velocity in the boundary layer. Eqs. (5) and (6) are presumed to be more accurate than any prior expressions in the literature, but numerical integration of Eqs. (2) and (3) is recommended for even greater accuracy. Eqs. (5) and

(6) may be extended speculatively for naturally rough plates by dividing the argument of the logarithmic terms by $[1+0.301(e/b)b^{+}]$. Predictions of u^{+} and $u_{\rm m}^{+}$ or f for specified values of $Re_{4b}=4b^{+}u_{\rm m}^{+}$ may readily be obtained by iterative solution of Eq. (6) with $u_{\rm m}^{+}$ replaced by $Re_{4b}/4b^{+}$ to determine the corresponding value of b^{+}.

Future improved experimental measurements or numerically computed values of $(\overline{u'v'})^{++}$, u^{+}, and/or f for parallel plates may support the modification of the coefficients 0.7, 0.436, 6.95, 13.35, 6.824, 5.314, 4.615 and 155.3 and of the exponents of $-8/7$ and -3 of Eqs. (4)–(6), but the resulting changes in the predictions of these three expressions may be expected to be quite small.

3. Heat transfer

The time-averaged energy balance for steady fully developed convection in the turbulent flow of a Newtonian fluid with invariant physical properties and negligible viscous dissipation between parallel plates of unlimited extent may be written in correspondence to Eq. (1) as:

$$\frac{j}{j_{\rm w}}[1-(\overline{T'v'})^{++}]=\frac{{\rm d}T^{+}}{{\rm d}y^{+}} \tag{7}$$

where $T^{+}\equiv k(T_{\rm w}-T)(\tau_{\rm w}\rho)^{1/2}/\mu j_{\rm w}$ and $(\overline{T'v'})^{++}=\rho c\overline{T'v'}/j$ are defined so as to be analogous to u^{+} and $(\overline{u'v'})^{++}$. Even so, the behavior described by Eq. (7) differs significantly from that of Eq. (1) because $(\overline{T'v'})^{++}$ is a function of the Prandtl number, $Pr=c\mu/k$, as well as of $(\overline{u'v'})^{++}$ or of y^{+} and b^{+}, and $j/j_{\rm w}$ is not equal to $(\tau/\tau_{\rm w})=1-(y^{+}/b^{+})$ for any thermal boundary conditions. For convenience, Eq. (7) may be expressed as

$$\frac{j}{j_{\rm w}}=\left[1+\frac{Pr}{Pr_{\rm t}}\left(\frac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)\right]\frac{{\rm d}T^{+}}{{\rm d}y^{+}} \tag{8}$$

where by comparison of Eqs. (7) and (8),

$$\frac{Pr_{\rm t}}{Pr}\equiv\left(\frac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)\left(\frac{1-(\overline{T'v'})^{++}}{(\overline{T'v'})^{++}}\right) \tag{9}$$

Thus, $Pr_{\rm t}/Pr$ may be interpreted simply as a symbol representing the local ratio of the transport of momentum by turbulence to that by molecular motion, divided by the equivalent ratio for the transport of energy. The quantities involved in this definition of $Pr_{\rm t}/Pr$ are obviously all physically meaningful and directly measurable. It is unnecessary to invoke eddy diffusional models for transport by turbulence to obtain Eqs. (8) and (9). The advantage of Eq. (8) over Eq. (7) accrues from the relative invariance of $Pr_{\rm t}$ as compared to that of $(\overline{T'v'})^{++}$.

The results that follow are based on Eq. (8) together with Eqs. (2)–(4). It is obviously necessary to have expressions for $j/j_{\rm w}$ and $Pr_{\rm t}/Pr$. The first of these two ratios depends on the thermal boundary conditions, whereas the latter does not. Accordingly, $Pr_{\rm t}/Pr$ will be considered separately in advance.

3.1. Prediction of the turbulent Prandtl number

Abbrecht and Churchill [1] demonstrated for a round tube that $Pr_{\rm t}$ is independent of the thermal boundary condition at the wall by determining this quantity experimentally for developing convection in fully developed flow and further found that their values of $Pr_{\rm t}$ were in agreement with those from measurements of heat transfer across a parallel-plate channel for $b^{+}=a^{+}$. The latter observation may be recognized to be in accord with the analogy of MacLeod [12]. These findings are also in accord with an expression derived by Yahkot et al. [14], using renormalization group theory, that relates $Pr_{\rm t}$ to Pr and $(\overline{u'v'})^{++}$ only and is thereby implied to be valid for all geometries and thermal boundary conditions.

Experimental data for $Pr_{\rm t}$ in the turbulent core for $Pr\geq 0.7$ have been correlated by Jischa and Rieke [10] and others by means of empirical expressions such as

$$Pr_{\rm t}=0.85+\frac{0.015}{Pr} \tag{10}$$

but attempts at generalized correlation for the region near the wall and for low-Prandtl-number fluids have been less successful. The recent computations of the equivalent of $Pr_{\rm t}$ by Papavassiliou and Hanratty [13] by both Eulerian and Lagrangian direct numerical simulations are in fair accord with Eq. (10) and at least quantitatively, with one significant exception, with the predictions of Yahkot et al. [14]. That exception is near the wall for asymptotically large values of Pr. For large values of Pr, the best experimental results for the heat transfer coefficient, which are limited to $Pr<100$, imply and the expression of Yahkot et al. [14] predicts the approach to a finite limiting value of $Pr_{\rm t}$ as $y^{+}\rightarrow 0$. On the other hand, the best experimental results for the rate of electrochemically driven mass transfer, which extend to much larger values of Sc, imply and the calculations of Papavassiliou and Hanratty [13] predict an indefinite increase in $Pr_{\rm t}$ as $y^{+}\rightarrow 0$ for very large values of Pr. This uncertainty in the turbulent Prandtl number precludes the calculation of definitive values of the Nusselt number in general at the present time. Fortunately, as described below, the effect of this uncertainty may be avoided or minimized for most conditions of practical interest.

3.2. Unequal uniform temperatures on the plates

This thermal boundary condition has been widely used both experimentally and computationally because of the resulting simplicity of the behavior, namely the existence of (or experimentally at least the close approach to) a uniform heat flux density across the channel. Furthermore, for fully developed convection, this thermal boundary condition is equivalent to that for one uniformly heated and one equally cooled plate. Then for $j = j_w$, Eq. (8) may be integrated formally to obtain

$$T^+ = \int_0^{y^+} \frac{dy^+}{1 + \frac{Pr}{Pr_t}\left(\frac{(\overline{u'v'})^{++}}{1 - (\overline{u'v'})^{++}}\right)} \tag{11}$$

for which it follows that

$$Nu_b = \frac{b^+}{T_b^+} = \frac{1}{\int_0^1 \frac{dZ}{1 + \frac{Pr}{Pr_t}\left(\frac{(\overline{u'v'})^{++}}{1 - (\overline{u'v'})^{++}}\right)}} \tag{12}$$

For both laminar flow, for which $(\overline{u'v'})^{++} = 0$, and turbulent flow in the limit of $Pr = 0$, Eq. (12) reduces to $Nu_b = 1$.

Insofar as $Pr = Pr_t$ for all y^+, Eq. (11) reduces to

$$T^+ = \int_0^{y^+} [1 - (\overline{u'v'})^{++}]dy^+ \tag{13}$$

and Eq. (12) to

$$Nu_b = \frac{1}{\int_0^1 [1 - (\overline{u'v'})^{++}]dZ} = \frac{1}{[1 - (\overline{u'v'})^{++}]_m} \tag{14}$$

Thus, Nu_b for $Pr = Pr_t$ is simply equal to the reciprocal of the integrated-mean value of $[1 - (\overline{u'v'})^{++}]$ over the cross-section. Values of Nu_b calculated from Eq. (14) using $(\overline{u'v'})^{++}$ from Eq. (4) are listed in Table 1 for a series of values of b' under the heading $Pr = 0.867$, which, according to Eq. (10), corresponds to $Pr = Pr_t$. Both experimental data and the predictive equation of Yahkot et al. [14] provide support for the invariance of Pr_t across the entire channel for nearly the same value of Pr.

For sufficiently larger values of Pr, the development of the temperature field is essentially confined to the region very near the wall, where, in accordance with the first term of Eq. (4),

$$(\overline{u'v'})^{++} \cong 0.7\left(\frac{y^+}{10}\right)^3 \tag{15}$$

Then, insofar as Pr_t may be postulated to be invariant in this regime, the integral of Eq. (11) may be carried out analytically to obtain

$$T^+ = \frac{10}{(0.7)^{1/3}\left(\frac{Pr}{Pr_t} - 1\right)^{4/3}}\left[\frac{Pr}{3\,Pr_t}\left(\frac{1}{2}\ln\left\{\frac{(1+z)^2}{1 - z + z^2}\right\} + 3^{1/2}\tan^{-1}\left\{\frac{2z-1}{3^{1/2}}\right\} + \frac{3^{1/2}\pi}{6}\right) - z\right] \tag{16}$$

where

$$z = (0.7)^{1/3}\left(\frac{Pr}{Pr_t} - 1\right)^{1/3}\left(\frac{y^+}{10}\right)$$

For large values of Pr/Pr_t and a sufficiently large value of y^+ (say 11), Eq. (16) approaches

$$T_\infty^+ = \frac{20\pi\left(\frac{Pr}{Pr_t}\right)}{3^{3/2}(0.7)^{1/3}\left(\frac{Pr}{Pr_t} - 1\right)^{4/3}} \tag{17}$$

from which it follows that

$$Nu_b = 0.07343\left(\frac{Pr_t}{Pr}\right)\left(\frac{Pr}{Pr_t} - 1\right)^{4/3} Re_b\left(\frac{f}{2}\right)^{1/2} \tag{18}$$

For $Pr \gg Pr_t = 0.85$, which follows from Eq. (10),

$$Nu_b \to 0.07752 Pr^{1/3}\, Re_b\left(\frac{f}{2}\right)^{1/2} \tag{19}$$

As inferred from the implicit independence of the heat transfer coefficient from the characteristic dimension, Eqs. (16)–(19) are applicable for all channels. Also, it may be inferred from the derivation that these expressions are applicable for fully developed convection with any thermal boundary condition at $y^+ = 0$.

Eqs. (11) and (12) are exact, and, insofar as the postulated independence of Pr_t from y^+ is valid, Eqs. (13) and (14) and (15)–(18) as well.

Eq. (10) and other empirical expressions for Pr_t have generally been based on the most reliable experimental data for this quantity, namely the values determined for the turbulent core for fluids with $0.7 < Pr < 100$. Nevertheless, this expression might be expected to provide an adequate approximation for the prediction of T^+ and Nu_{2b} for all values of Pr since the term

$$\frac{Pr}{Pr_t}\left(\frac{(\overline{u'v'})^{++}}{1 - (\overline{u'v'})^{++}}\right)$$

in Eqs. (11) and (12) is small with respect to unity near the wall by virtue of small values of $(\overline{u'v'})^{++}$ and for $Pr < 0.7$ for all y^+ by virtue of small values Pr/Pr_t. That is, the effect of Pr_t in the two regimes in which it is most uncertain is dampened within the integrand and then further by the integration itself. Calculated values of Nu_b for a wide range of values of Pr, based on the use of Pr_t from Eq. (10) in Eq. (12), are included in Table 1. Those for large values of Pr are expressed as $Nu_b/0.07752 Pr^{1/3}\, Re_b (f/2)^{1/2}$ to restrain their magnitude and reveal their approach to the asymptotic expression for $Pr \to \infty$. As would be expected the approximate values are consistent with the more certain ones for $Pr = 0$, 0.867 and ∞.

3.3. Equal uniform heating from both plates

Before deriving expressions for the temperature distribution and the Nusselt number for equal uniform heating from both walls of the channel, it is necessary to have an expression for j/j_w as well as for $(\overline{u'v'})^{++}$ and Pr_t. For fully developed symmetrical convective heating of a fluid in either laminar or turbulent flow between parallel plates, it may be shown from a segmental energy balance that

$$\frac{j}{j_w} = \frac{1}{b^+}\int_{y^+}^{b^+} \frac{u^+}{u_m^+}\left(\frac{\partial T/\partial x}{\partial T_m/\partial x}\right) dy^+ \tag{20}$$

For equal uniform heating on both plates, $\partial T/\partial x = \partial T_m/\partial x$, and Eq. (20) reduces to

$$\frac{j}{j_w} = \frac{1}{b^+}\int_{y^+}^{b^+} \left(\frac{u^+}{u_m^+}\right) dy^+ = \int_0^Z \left(\frac{u^+}{u_m^+}\right) dZ \tag{21}$$

Before determining the behavior of j/j_w in general, it is useful to examine four limiting cases. Thus, for the hypothetical case of plug flow, Eq. (21) reduces to

$$\frac{j}{j_w} = 1 - \frac{y^+}{b^+} = Z \tag{22}$$

while for laminar flow

$$\frac{j}{j_w} = \frac{Z}{2}(3 - Z^2) \tag{23}$$

For very small values of y^+, such that Eq. (15) is applicable,

$$u^+ \to y^+ - \frac{7}{4}\left(\frac{y^+}{10}\right)^4 - \frac{(y^+)^2}{2b^+} \ldots \tag{24}$$

Table 1
Predicted Nusselt numbers for fully developed turbulent convection between isothermal plates[a]

Nu_b for small values of Pr

b^+	$Re_b \times 10^{-3}$	Pr						
		0	0.001	0.01	0.10	0.70	0.867	1.0
< 60	< 0.65	1.0	1.0	1.0	1.0	1.0	1.0	1.0
500	9.279	1.0	1.002	1.113	3.337	14.164	16.434	18.108
1000	20.312	1.0	1.003	1.230	5.609	26.739	31.196	34.490
5000	120.66	1.0	1.018	2.126	21.857	118.796	139.758	155.321
10,000	257.36	1.0	1.036	3.186	40.535	226.870	267.593	297.970
50,000	1471.4	1.0	1.175	10.834	175.258	1027.90	1218.26	1360.650

$Nu_b/0.07752 Pr^{1/3}\, Re_b (f/2)^{1/2}$ for large values of Pr

b^+	$Re_b \times 10^{-3}$	Pr						
		1.0	10	100	1000	10,000	25,000	∞
500	9.279	0.4672	0.7973	0.9431	0.9858	0.9979	0.9990	1.000
1000	20.312	0.4449	0.7848	0.9396	0.9821	0.9967	0.9989	1.000
5000	120.66	0.4007	0.7550	0.9288	0.9796	0.9962	0.9977	1.000
10,000	257.36	0.3844	0.7426	0.9277	0.9789	0.9954	0.9978	1.000
50,000	1471.4	0.3510	0.7177	0.9258	0.9782	0.9922	0.9984	1.000

[a] Values of Re_b based on Eq. (6).

and from Eq. (21)

$$\frac{j}{j_w} \to 1 - \frac{(y^+)^2}{2b^+ u_m^+}\left[1 - 0.07\left(\frac{y^+}{10}\right)^3 - \frac{y^+}{3b^+}\ldots\right] \tag{25}$$

For the other limit of $y^+ \to b^+$, it may be inferred that

$$\frac{j}{j_w} \to \left(1 - \frac{y^+}{b^+}\right)\frac{u_b^+}{u_m^+} = \frac{Zu_b^+}{u_m^+} \tag{26}$$

For intermediate values of y^+, it is preferable in terms of accuracy to re-express Eq. (21) in terms of $(\overline{u'v'})^{++}$ by means of Eqs. (2) and (3), resulting in

$$\frac{j}{j_w} = \frac{\dfrac{1}{b^+}\displaystyle\int_{y^+}^{b^+}\left(\int_0^{y^+}[1-(\overline{u'v'})^{++}]\left(1-\frac{y^+}{b^+}\right)\mathrm{d}y^+\right)\mathrm{d}y^+}{\displaystyle\int_0^{b^+}[1-(\overline{u'v'})^{++}]\left(1-\frac{y^+}{b^+}\right)^2\mathrm{d}y^+} \tag{27}$$

which may be reduced by integration by parts to

$$\frac{j}{j_w} = \frac{\left(1-\dfrac{y^+}{b^+}\right)\displaystyle\int_0^{y^+}[1-(\overline{u'v'})^{++}]\left(1-\frac{y^+}{b^+}\right)\mathrm{d}y^+ + \int_{y^+}^{b^+}[1-(\overline{u'v'})^{++}]\left(1-\frac{y^+}{b^+}\right)^2\mathrm{d}y^+}{\displaystyle\int_0^{b^+}[1-(\overline{u'v'})^{++}]\left(1-\frac{y^+}{b^+}\right)^2\mathrm{d}y^+} \tag{28}$$

Eq. (28) may be noted to reduce to the equivalent of Eq. (26) in terms of $(\overline{u'v'})^{++}$ as $y^+ \to b^+$.

In the interests of accuracy and convenience in interpretation, j/j_w may be replaced as a variable by a perturbation γ defined by

$$\frac{j}{j_w} = \left(1-\frac{y^+}{b^+}\right)(1+\gamma) \tag{29}$$

Thus, γ represents the deviation from the linear behavior for plug flow as given by Eq. (22).

Values of γ computed from Eqs. (28) and (29) using $(\overline{u'v'})^{++}$ from Eq. (4) are listed in Table 2 for several values of b^+ and y^+ or y^+/b^+. The deviation from linearity, as represented by the magnitude of γ, is seen to be significant near the central plane. The neglect of this quantity or the even more erroneous postulate of $j=j_w$ are the major sources of error in many prior theoretical results.

Eq. (8), after introduction of γ from Eq. (27), may be expressed in the following integral form:

$$T^+ = \int_0^{y^+}\frac{(1+\gamma)\left(1-\dfrac{y^+}{b^+}\right)\mathrm{d}y^+}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)} \tag{30}$$

It follows from Eqs. (21) and (30) and integration by parts that

$$T_m^+ \equiv \frac{1}{b^+}\int_0^{b^+}T^+\left(\frac{u^+}{u_m^+}\right)\mathrm{d}y^+$$

$$= \int_0^{b^+}\frac{(1+\gamma)^2\left(1-\dfrac{y^+}{b^+}\right)^2\mathrm{d}y^+}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)} \tag{31}$$

Then

$$Nu_{4b} = \frac{4b^+}{T_m^+} = \frac{4}{\displaystyle\int_0^1\frac{(1+\gamma)^2\left(1-\dfrac{y^+}{b^+}\right)^2\mathrm{d}\left(\dfrac{y^+}{b^+}\right)}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)}} = \frac{12}{\displaystyle\int_0^1\frac{(1+\gamma)^2\,\mathrm{d}Z^3}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)}} \tag{32}$$

The choice of 4*b* as the characteristic dimension is in accord with the conventional use of the hydraulic diameter. The alternative choice of *b* for heat transfer across a channel has the obvious objective of resulting in $Nu_b = 1$ in laminar flow and for $Pr = 0$.

For $Pr = 0$, Eq. (30) reduces to

$$T^+ = \int_0^{y^+} (1+\gamma)\left(1 - \frac{y^+}{b^+}\right) dy^+ \tag{33}$$

and Eq. (32) to

$$Nu_{4b} = \frac{12}{\int_0^1 (1+\gamma)^2 \, dZ^3} = \frac{12}{(1+\gamma)^2_{mZ^3}} \tag{34}$$

where $(1+\gamma)^2_{mZ^3}$ symbolizes the integrated-mean value of $(1+\gamma)^2$ with respect to Z^3. Using γ from Eq. (23) gives $Nu_{4b} = 140/17 = 8.235$ for laminar flow, while the values of γ from Eq. (28) lead to the values of Nu_{4b} listed in Table 3 for $Pr = 0$.

For $Pr = Pr_t$, Eq. (30) reduces to

$$T^+ = \int_0^{y^+} (1+\gamma)[1 - (\overline{u'v'})^{++}]\left(1 - \frac{y^+}{b^+}\right) dy^+ \tag{35}$$

which differs from Eq. (2) for u^+ only by virtue of the factor $(1+\gamma)$ in the integrand. Because of the difference between the shear stress ratio, τ/τ_w, and the heat flux density ratio, j/j_w, the dimensionless temperature distribution differs from the dimensionless velocity distribution even for $Pr = Pr_t$. Eq. (32), on the other hand, reduces to

$$Nu_{4b} = \frac{12}{\int_0^1 (1+\gamma)^2[1 - (\overline{u'v'})^{++}] dZ^3} \tag{36}$$

The factor $(1+\gamma)^2$ in Eq. (36) represents the combined effects of the heat flux density distribution and the time-averaged velocity distribution. Values of Nu_{4b} computed from Eq. (36) using $(\overline{u'v'})^{++}$ from Eq. (4) and $(1+\gamma)$ from Eq. (28) are included in Table 3 under the heading $Pr = 0.867$.

Eq. (36) when rewritten as

$$Nu_{4b} = \frac{12}{\left(\dfrac{\int_0^1 (1+\gamma)^2[1 - (\overline{u'v'})^{++}] dZ^3}{\int_0^1 [1 - (\overline{u'v'})^{++}] dZ^3}\right)\left(\int_0^1 [1 - (\overline{u'v'})^{++}] dZ^3\right)} \tag{37}$$

may be recognized by virtue of Eq. (3) as exactly equivalent to

$$Nu_{4b} = \frac{Re_{4b}\left(\dfrac{f}{2}\right)}{(1+\gamma)^2_{wmZ^3}} \tag{38}$$

where $(1+\gamma)^2_{wmZ^3}$ symbolizes the integrated-mean value of $(1+\gamma)^2$, weighted by $[1 - (\overline{u'v'})^{++}]$, with respect to Z^3. Thus, the Reynolds analogy is applicable functionally for $Pr = Pr_t \cong 0.867$ rather than for

Table 2
Computed values of γ

y^+	y^+/b^+	b^+					
		< 60	500	1000	5000	10,000	50,000
0		0	0	0	0	0	0
1		–	0.0019555	0.0009788	0.0001963	0.0000982	0.0000197
5		–	0.0087835	0.0044252	0.0009003	0.0004531	0.0000918
10		–	0.0153083	0.0077876	0.0016167	0.0008193	0.0001684
20		–	0.0245231	0.0127000	0.0027342	0.0014038	0.0002954
	0.1	0.095	0.0436646	0.0370390	0.0290813	0.0270209	0.0234252
	0.2	0.180	0.0679830	0.0591629	0.0477073	0.0444846	0.0386064
	0.3	0.255	0.0883069	0.0777222	0.0633341	0.0591335	0.0512598
	0.4	0.320	0.1060406	0.0939226	0.0769753	0.0719213	0.0622138
	0.5	0.375	0.1215199	0.1080674	0.0888811	0.0830525	0.0722107
	0.6	0.420	0.1347398	0.1201462	0.0990406	0.0925913	0.0801369
	0.7	0.455	0.1455271	0.1300031	0.1073300	0.1003501	0.0860703
	0.8	0.480	0.1536278	0.1374031	0.1135108	0.1061109	0.0911335
	0.9	0.495	0.1587318	0.1420678	0.1174209	0.1097183	0.0940324
	1.0	0.500	0.1605235	0.1437123	0.1189158	0.1112922	0.0974044

Pr = 1, and even then is in numerical error by the indicated factor. However, the computed values of $(1+\gamma)^2_{\mathrm{wm}Z^3} = 16(b^+)^2/Re_{4b}\ Nu_{4b}\{0.867\}$ in Table 3 may be observed to differ only slightly from unity owing to the partial compensation of the local values of $(1+\gamma)^2$ and $[1-(\overline{u'v'})^{++}]$. Eqs. (18) and (19) with Nu_{4b} and Re_{4b} replacing Nu_b and Re_b are directly applicable for large Pr and $Pr \to \infty$, respectively.

Eqs. (30)–(34) may be considered to be exact and, insofar as the postulated independence of Pr_t from y^+ is valid, Eqs. (35)–(38) and the adapted versions of Eqs. (18) and (19) as well.

For the same reasons as discussed in connection with heat transfer between the plates, the use of Pr_t from Eq. (10) in Eq. (32) might be expected to provide a reasonable approximation for Nu_{2b} for all values of Pr. Values of Nu so-calculated for a series of values of Pr and b^+ are included in Table 3. Those for large values of Pr are expressed in terms of $Nu_{4b}/0.07752 Pr^{1/3}\ Re_{4b}(f/2)^{1/2}$ in order to reduce their magnitude and reveal their approach to the asymptotic expression for $Pr \to \infty$. The approximate values in Table 3 for Pr = 0.001, 0.01, 0.1, 0.7, 1.0, 10, 100, 1000, 10,000 and 25,000 appear to be consistent with the more certain values for Pr = 0, 0.867 and ∞, just as they were for the other mode of heating.

3.4. Prior values

Kays and Leung [11] derived perhaps the most accurate prior values for convection in turbulent flow between parallel plates. They solved by a finite-difference method the equivalent of Eq. (8) expressed in terms of the time–mean velocity rather than j/j_w, and the eddy viscosity rather than the local turbulent shear stress. They utilized empirical expressions for the turbulent Prandtl number and separate and thereby incongruent expressions for the time–mean velocity and the eddy viscosity. These results are for $Re_{4b} = 10^4$, 3×10^4, 10^5, 3×10^5 and 10^6, a series of values of Pr from 0 to 1000, and arbitrary ratios of uniform heating on the two plates. Some of these values for Nu_{4b} for equal uniform heating with Pr = 0, Pr = 0.867 (as interpolated between Pr = 0.7 and 1.0) and Pr = 1000 are compared in Table 4 with values obtained from the essentially exact predictions herein. The small differences for Pr = 0 are presumed to be a consequence of the improved velocity distribution utilized in the current work since the turbulent Prandtl number and eddy viscosity phase out in this limit. The somewhat greater discrepancies for Pr = 0.867 and 1000 are presumed to be a consequence of improvements in the representation of the rate of turbulent transport, as well as the velocity distribution in the work herein.

Table 3
Predicted Nusselt numbers for fully developed convection in turbulent flow between uniformly heated plates[a]

Nu_{4b} for small values of Pr

b^+	$Re_{4b} \times 10^{-3}$	Pr						$16b^{+2}/Re_{4b}$	$(1+\gamma)^2_{\mathrm{wm}Z^3}$
		0	0.001	0.01	0.10	0.70	0.867		
< 60	< 2.60	8.235	8.235	8.235	8.235	8.235	8.235	12.0000	1.4572
500	37.116	10.432	10.448	11.465	28.931	90.400	101.34	107.770	1.0634
1000	81.249	10.609	10.642	12.758	46.659	166.28	187.95	196.926	1.0478
5000	482.64	10.854	11.026	21.154	162.428	701.57	804.73	828.775	1.0299
10,000	1029.5	10.927	11.272	30.312	288.010	1314.42	1515.26	1554.21	1.0257
50,000	5885.5	11.066	12.777	89.955	1141.89	5731.60	6667.95	6796.33	1.0193

$Nu_{4b}/0.07752 Re_{4b}(f/2)^{1/2}\ Pr^{1/3}$ for large values of Pr

b^+	$Re_{4b} \times 10^{-3}$	Pr					
		1.0	10	100	1000	10,000	25,000
500	37.116	0.68999	0.84005	0.97529	0.99005	0.99852	0.99951
1000	81.249	0.63973	0.82007	0.96726	0.98758	0.99795	0.99937
5000	482.64	0.56516	0.80005	0.96255	0.98503	0.99795	0.99899
10,000	1029.5	0.52146	0.77062	0.95988	0.98251	0.99777	0.99890
50,000	5885.5	0.46285	0.74997	0.95487	0.98096	0.99756	0.99808

[a] Values of Re_{4b} based on Eq. (6).

Table 4
Comparison of predicted values of Nu_{4b} for fully developed turbulent convection from two uniformly heated plates[a]

b^+	$Re_{4b} \times 10^{-3}$	Nu_{4b}				$Nu_{4b}/Pr^{1/3}$	
		$Pr = 0$		$Pr = 0.867$		$Pr = 1000$	
		K&L	Herein	K&L	Herein	K&L	Herein
415	30	10.41	10.40	84.90	85.90	99.90	127.45
1204	100	10.66	10.66	212.3	222.0	288.60	368.62
3244	300	10.74	10.81	514.2	543.1	766.50	991.92
9737	1000	10.90	10.93	1392.0	1478.0	2305.0	2966.9

[a] K&L, Kays and Leung [11]; b^+ based on specified values of Re and Eq. (6).

4. Conclusions

The integral expressions derived and utilized herein for u^+ and $u_m^+ = (2/f)^{1/2}$ are exact for invariant physical properties insofar as fully developed flow is attained. The rate of transport of momentum predicted by Eq. (4) is presumed to be more accurate than any prior expressions in terms of the eddy viscosity or the mixing length. The smoothing that results from evaluation of the integrals in Eqs. (2) and (3) reduces the slight inaccuracy of Eq. (4) and results in even more accurate values of u^+ and $u_m^+ = (2/f)^{1/2}$. Such values are presumed to be an improvement on all prior theoretical results, at least for $b^+ > 500$ ($Re_{4b} > 37{,}116$). Although some additional error is necessarily introduced by the empirical correlation of these computed values by Eqs. (5) and (6), the later expressions are nevertheless presumed to be more accurate than any prior correlating equations (those based on experimental values are free from an erroneous expression for j/j_w but necessarily reflect any non-random errors in the measurements as well as those of form).

The corresponding integral expressions for T^+ and Nu, namely Eqs. (11) and (12) for heat transfer between the plates, and Eqs. (30) and (32) for heat transfer to the fluid stream, are also exact for invariant properties insofar as fully developed convection is attained and viscous dissipation is negligible. The factor $1+\gamma$ that appears in the integrand for heating from both plates introduces negligible error because of the smoothing that results from the integration of $(\overline{u'v'})^{++}$ by means of which it is evaluated, but the current uncertainty in the turbulent Prandtl results in an associated uncertainty in the computed values of the Nusselt number for both thermal boundary conditions. Hence, the numerical results presented herein are subject to future improvement when the dependence of the turbulent Prandtl number on Pr and $(\overline{u'v'})^{++}$ or y^+ and b^+ becomes known with greater certainty. Even so, the values predicted using Eqs. (12) and (32) with $(\overline{u'v'})^{++}$ from Eq. (4) and Pr_t from Eq. (10), and in the latter case with γ from Eqs. (28) and (29) are presumed to be more accurate than any prior predictions for $Pr = 0$ and $Pr = 0.867$ and probably for all other values of Pr as well.

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